A THOM-SMALE-WITTEN THEOREM ON MANIFOLDS WITH BOUNDARY

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Abstract. Given a smooth compact manifold with boundary, we show that the subcomplex of the deformed de Rham complex consisting of eigenspaces of small eigenvalues of the Witten Laplacian is canonically isomorphic to the Thom-Smale complex constructed by Laudenbach in [12]. Our proof is based on Bismut-Lebeau's analytic localization techniques. As a by-product, we obtain Morse inequalities for manifolds with boundary.

1. Introduction

Morse theory is a method to determine the topology of a finite or infinite dimensional manifold (such as the space of paths or loops on a compact manifold) from the critical points of one suitable function on the manifold. The theory has many far-reaching applications ranging from existence of exotic spheres to supersymmetry and Yang-Mills theory.

In this paper we will study Morse theory on a manifold with boundary. The main goal is to exhibit a canonical isomorphism between the Witten instanton complex constrained to boundary conditions and the Thom-Smale complex.

Recall that a Morse function on a manifold without boundary is a smooth real function whose critical points are all non-degenerate. Let M be a smooth n-dimensional closed manifold. Let f be a Morse function on M and choose a Riemannian metric on M such that the gradient vector field ∇f satisfies the Morse-Smale transversality conditions. This dynamical system gives rise to a chain complex, called Thom-Smale complex, having the critical points as generators and boundary map expressed in terms of the unstable and stable manifolds. The Morse homology theorem asserts that the Thom-Smale complex recovers the standard homology of the underlying manifold (see [11, 16] and the colourful historical presentation [5]).

In [19] Witten suggested that the Thom-Smale complex could be recovered from the Witten instanton complex, which is subcomplex of the deformed de Rham complex, consisting of eigenspaces of small eigenvalues of the Witten Laplacian D_T^2 (cf. (4.5)). This fact was first established rigorously by Helffer and Sjöstrand [10, Prop. 3.3].

Later, Bismut and Zhang [2] generalized the results of Helffer and Sjöstrand for the Witten instanton complex with values in a flat vector bundle. This is an important step in their proof and extension of the Cheeger-Müller theorem [8, 14] about the equality of the Reidemeister and Ray-Singer metrics.

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In [3, §6], Bismut and Zhang gave a new and simple proof of the isomorphism between the Witten instanton complex and the Thom-Smale complex by means of resolvent estimates. We refer the readers to [20, Chapter 6] for a comprehensive study of the Witten deformation following Bismut-Zhang's approach.

Let us go back to the general case of a compact (not necessarily orientable) manifold M of dimension n with boundary $\partial M \neq \emptyset$. In this paper, a smooth function $f : M \to \mathbb{R}$ is called a Morse function if the restrictions of f to the interior and boundary of M are Morse functions in the usual sense and if f has no critical point on ∂M (cf. [7, 12]).

In order to have Hodge theory for the de Rham Laplacian, we will impose boundary conditions: absolute boundary condition and relative boundary condition (cf. [18, pp. 361-371]). Denote by $\mathcal{H}_a$ (resp. $\mathcal{H}_r$) the subspace of harmonic forms satisfying the absolute boundary condition (resp. the relative boundary condition). Then the space $\mathcal{H}_a$ (resp. $\mathcal{H}_r$) is isomorphic to the absolute cohomology group $H^\bullet_{dR}(M, \mathbb{R})$ (resp. the relative cohomology group $H^\bullet_{dR}(M, \partial M; o(TM))$ with coefficients twisted by the orientation bundle $o(TM)$ of M).

In [7], Chang and Liu established the Morse inequalities corresponding to the absolute and relative boundary conditions for orientable manifolds, by using Witten deformation.

On the other hand, Laudenbach [12] recently constructed a Thom-Smale complex whose homology is isomorphic to the (absolute or relative) homology of M with integral coefficients. The construction uses a pseudo-gradient Morse-Smale vector field suitably adapted to the boundary.

We will prove that the Witten instanton complex constrained to the boundary conditions and the Thom-Smale complex constructed by Laudenbach in [12] are canonically isomorphic. This result is new for manifolds with non-empty boundary, while its counterpart for closed manifolds appeared in [2, 10]. Our method consists of applying Bismut-Lebeau's localization techniques [4] along the lines of [20].

In order to state the results let us introduce some notations. Let $f : M \to \mathbb{R}$ be a Morse function on M, and let $f|_{\partial M}$ be its restriction to the boundary. Let $C^j(f)$ (resp. $C^j(f|_{\partial M})$ be the set consisting of all critical points of f (resp. $f|_{\partial M}$) with index j. Let ν be the outward normal vector field along ∂M. Denote by

$$C^j_-(f|_{\partial M}) = \{p \in C^j(f|_{\partial M}) : (\nu f)(p) < 0\}\,, \quad C^j_+(f|_{\partial M}) = \{p \in C^j(f|_{\partial M}) : (\nu f)(p) > 0\}\,,$$

and

$$c_j = \#\, C^j(f)\,, \quad p_j = \#\, C^j_-(f|_{\partial M})\,, \quad q_j = \#\, C^j_+(f|_{\partial M})\,.$$

Set $C(f) = \bigsqcup_{j=1}^n C^j(f)$ and $C_-(f|_{\partial M}) = \bigsqcup_{j=0}^{n-1} C^j_-(f|_{\partial M})$.

In order to define the Witten instanton complex we have to state two results about the spectrum of the Witten Laplacian. Let

$$0 \leqslant \lambda^j_1(T) \leqslant \lambda^j_2(T) \leqslant \cdots \tag{1.1}$$

be the eigenvalues of the Witten Laplacian D^2_T with absolute boundary condition acting on j-forms.

Theorem 1.1. *There exists positive constants a_1 and T_1 such that for $T > T_1$, and $j = 0, 1, \cdots, n$, we have*

$$\lambda_\ell^j(T) \geqslant a_1 T^2, \quad \text{for } \ell \geqslant c_j + p_j + 1. \tag{1.2}$$

The estimate (1.2) was obtained by Chang and Liu (cf. [7, §3, Th. 2]) by localization and the min-max principle and by Le Peutrec (cf. [15, Th. 3.5]) by delicate constructions of quasimodes and the WKB method. We will give here a short proof based on elementary spectral estimates.

Our second result provides a refined estimate of the lower part of the spectrum of the Witten Laplacian.

Theorem 1.2. *There exist positive constants a_2, a_3 and T_2 such that for $T \geqslant T_2$ and $j = 0, 1, \cdots, n$,*

$$\lambda_\ell^j(T) \leqslant a_2 e^{-a_3 T}, \quad \text{for } \ell \leqslant c_j + p_j. \tag{1.3}$$

The estimate (1.3) for $j = 0$ was obtained by D. Le Peutrec via WKB analysis (cf. [15, Th. 1.0.3]).

From Theorems 1.1 and 1.2 follows that the Witten Laplacian D_T^2 acting on j-forms has a spectral gap: the upper part of the spectrum grows with speed T^2 and the lower part decays exponentially in T. Moreover, for a given $C_0 > 0$, there exists $T_0 > 0$ such that for $T \geqslant T_0$, the number of eigenvalues in $[0, C_0)$ equals $c_j + p_j$ (see Proposition 4.1). Let $F_{T,j}^{C_0}$ denote the $(c_j + p_j)$-dimensional vector space generated by the eigenspaces associated to the eigenvalues lying in $[0,\ C_0)$. It is easy to see that $(F_{T,\bullet}^{C_0}, d_T)$ forms a complex, called Witten instanton complex.

Let $(C^\bullet, \partial)$ denote the Thom-Smale complex constructed in [12] (cf. (2.7)–(2.8)).

Let P_∞ be the natural morphism from the de Rham complex $(\Omega^\bullet(M), d)$ to the Thom-Smale complex $(C^\bullet, \partial)$, defined by integration on the closure of the unstable manifolds (cf. §2):

$$P_\infty(\alpha) = \sum_{p \in C(f) \cup C_-(f|_{\partial M})} [p]^* \int\limits_{\overline{W^u(p)}} \alpha \in C^\bullet, \quad \text{for } \alpha \in \Omega^\bullet(M)\,. \tag{1.4}$$

Set

$$P_{\infty,T}(\alpha) = P_\infty(e^{Tf}\alpha), \text{ for } \alpha \in F_{T,\bullet}^{C_0}\,. \tag{1.5}$$

The main result of this paper is as follows.

Theorem 1.3. *The map $P_{\infty,T} : (F_{T,\bullet}^{C_0}, d_T) \to (C^\bullet, \partial)$ is an isomorphism of complexes for T large enough. In particular, the chain map P_∞ is a quasi-isomorphism between the de Rham complex $(\Omega^\bullet(M), d)$ and the Thom-Smale complex $(C^\bullet, \partial)$.*

Due to the de Rham isomorphism $H_{dR}^\bullet(M, \mathbb{R}) \cong H^\bullet(M, \mathbb{R})$ between de Rham cohomology and singular cohomology, we have thus an analytic proof of the fact that the cohomology of the Thom-Smale complex $(C^\bullet, \partial)$ coincides with the singular cohomology of M. This was proved by topological methods by Laudenbach [12].

Let $\beta_j(M)$ denote the j-th Betti number of the de Rham complex, i.e., $\beta_j(M) = \dim H^j_{dR}(M, \mathbb{R})$. As a by-product of the proof of Theorem 1.3, we obtain the following Morse inequalities for manifolds with boundary:

Corollary 1.4. *For any $k = 0, 1, \cdots, n$, we have*

$$\sum_{j=0}^{k} (-1)^{k-j} \beta_j(M) \leqslant \sum_{j=0}^{k} (-1)^{k-j} (c_j + p_j), \tag{1.6}$$

with equality for $k = n$.

The Morse inequalities (1.6) were first established in [7, §4]. A topological proof of (1.6) appeared in [12]. Denote by $\beta_j(M, \partial M)$ the j-th Betti number of the relative de Rham complex with coefficients twisted by the orientation bundle, i.e., $\beta_j(M, \partial M) = \dim H^j_{dR}(M, \partial M; o(TM))$. By replacing the Morse function f by $-f$ and applying the Poincaré duality theorem, we get from Corollary 1.4 the following result.

Corollary 1.5. *For any $k = 0, 1, \cdots, n$, we have*

$$\sum_{j=0}^{k} (-1)^{k-j} \beta_j(M, \partial M) \leqslant \sum_{j=0}^{k} (-1)^{k-j} (c_j + q_{j-1}), \tag{1.7}$$

with equality for $k = n$.

On the other hand, if we endow the Witten Laplacian D_T^2 with relative boundary condition, we will obtain analogue results of Theorems 1.1, 1.2 and 1.3. Then we also derive the inequalities (1.7). See Remark 5.7.

This paper is organized as follows. In Section 2, we introduce the Thom-Smale complex constructed by Laudenbach in [12]. Section 3 is devoted to some calculations of the kernels of the Witten Laplacian on Euclidean spaces. The results will be applied to the Witten Laplacian on manifolds around the critical points in $C(f)$ and $C_-(f|_{\partial M})$. In Section 4, we state a crucial result (Proposition 4.1) concerning the lower part of the spectrum of the Witten Laplacian for T large. We also prove Corollaries 1.4 and 1.5 there. In Section 5, we prove Proposition 4.1 and then finish the proof of Theorem 1.3.

2. The Thom-Smale complex constructed by Laudenbach

In this section, we introduce the Thom-Smale complex constructed by Laudenbach in [12]. By [12, §2.1], there exists a vector field X on M satisfying the following conditions.
(1) $(Xf)(\cdot) < 0$ except at critical points in $C(f) \cup C_-(f|_{\partial M})$;
(2) X points inwards along ∂M except in a neighborhood in ∂M of critical points in $C_-(f|_{\partial M})$ where it is tangent to ∂M;
(3) if $p \in C^j(f)$, then there exists a coordinate system (x, U_p) such that on U_p we have

$$f(x) = f(p) - \frac{x_1^2}{2} - \cdots - \frac{x_j^2}{2} + \frac{x_{j+1}^2}{2} + \cdots + \frac{x_n^2}{2}\,, \tag{2.1}$$

and

$$X = \sum_{i=1}^{j} x_i \frac{\partial}{\partial x_i} - \sum_{i=j+1}^{n} x_i \frac{\partial}{\partial x_i}\,; \tag{2.2}$$

(4) if $p \in C_-^j(f|_{\partial M})$, then there are coordinates $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ on some neighborhood U_p of p such that on U_p,

$$f(x) = f(p) - \frac{x_1^2}{2} - \cdots - \frac{x_j^2}{2} + \frac{x_{j+1}^2}{2} + \cdots + \frac{x_{n-1}^2}{2} + x_n \tag{2.3}$$

and

$$X = \sum_{i=1}^{j} x_i \frac{\partial}{\partial x_i} - \sum_{i=j+1}^{n} x_i \frac{\partial}{\partial x_i}; \tag{2.4}$$

(5) X is Morse-Smale in the sense that the global unstable manifolds and the local stable manifolds are mutually transverse. Denote by $W^u(p)$ (resp. $W^s_{loc}(p)$) the unstable manifold (resp. the local stable manifold) of p which by definition, consists of all the flow lines of X that emanate from p (resp. end at p). We denote by $\overline{W^u(p)}$ the closure of $W^u(p)$ in M.

If $p \in C^j(f)$, then we get from the condition (3) that locally,

$$W^u(p) = \Big\{(x_1, \cdots, x_j, 0, \cdots, 0)\Big\} \subset \mathbb{R}^n. \tag{2.5}$$

In view of the conditions (1) and (2), $W^u(p)$ does not intersect with the boundary except at critical points in $C_-(f|_{\partial M})$.

If $p \in C_-^j(f|_{\partial M})$, then the condition (4) implies that

$$W^u(p) = \Big\{(x_1, \cdots, x_j, 0, \cdots, 0)\Big\} \subset \mathbb{R}^{n-1} \times \mathbb{R}_+\,, \tag{2.6}$$

that is, $W^u(p)$ lies completely in the closed submanifold ∂M. Therefore, the results [11, Prop. 2] about the structure of $\overline{W^u(p)}$ still hold for $p \in C^j(f) \cup C_-^j(f|_{\partial M})$, i.e., $\overline{W^u(p)}$ is a j-dimensional submanifold of M with conical singularities and $\overline{W^u(p)} \backslash W^u(p)$ is stratified by unstable manifolds of critical points of index strictly less than j.

If $q \in C^{j+1}(f)$ and $p \in C^j(f)$ (resp. $q \in C^{j+1}(f)$ and $p \in C_-^j(f|_{\partial M})$, resp. $q \in C_-^{j+1}(f|_{\partial M})$ and $p \in C_-^j(f|_{\partial M})$), then it is a consequence of the condition (5) that the intersection of $W^u(q)$ with $W^s_{loc}(p)$ consists of a finite set of integral curves of X. Choose an orientation on $W^u(q)$ and $W^u(p)$, respectively. The orientation of $W^s_{loc}(p)$ is uniquely determined. Take $\gamma \in W^u(q) \cap W^s_{loc}(p)$. For $x \in \gamma$, denote by B_x the sets such that (X_x, B_x) forms an positive basis of $T_x\big(W^u(q)\big)$. Let A_x denote a positive basis of $T_x\big(W^s_{loc}(p)\big)$. Then (A_x, B_x) forms a basis of T_xM. Set $n_\gamma(q,p) = 1$ if (A_x, B_x) denotes a positive orientation of T_xM. Otherwise take $n_\gamma(q,p) = -1$. Let $n(q,p)$ be the sum of $n_\gamma(q,p)$ over all integral curves γ from q to p. Set $n(q,p) = 0$ if $q \in C_-^{j+1}(f|_{\partial M})$ and $p \in C^j(f)$.

For $p \in C^j(f) \cup C_-^j(f|_{\partial M})$, let $[p]$ be the real line generated by p, and let $[p]^*$ be the line dual to $[p]$. As in [12], set

$$C^j = \bigoplus_{p \in C^j(f) \cup C_-^j(f|_{\partial M})} [p]^*. \tag{2.7}$$

The boundary morphism ∂ from C^j to C^{j+1} is given by

$$\partial[p]^* = \sum_{q\in C^{j+1}(f)\cup C_-^{j+1}(f|_{\partial M})} n(q,p)[q]^*. \tag{2.8}$$

It is a consequence of [12, §2.2, Prop.] that $(C^\bullet, \partial)$ is a chain complex.

For $a > 0$, denote by $B^M(p, 4a)$ the open ball in M centered at the point $p \in M$ with the radius $4a$. In the sequel, we always take for simplicity $U_p = B^M(p, 4a)$ in the condition (2) and $U_p = B^M(p, 4a) \cap \partial M$ in the condition (3).

3. Some Calculations on Vector Spaces

In this section, we calculate the kernels of the Witten Laplacian on vector spaces. The results will be applied to the Witten Laplacian around the critical points in $C^j(f)$ and $C_-^j(f|_{\partial M})$.

3.1. **The Witten Laplacian on $\mathbb{R}^n$.** Let V be an n-dimensional real vector space endowed with an Euclidean scalar product. Let V^-, V^+ be two subspaces such that $V = V^- \oplus V^+$ and dim $V^- = j$. Take $e_1, \cdots, e_n$ as an orthonormal basis on V such that V^- is spanned by $e_1, \cdots, e_j$. Let f be a smooth function on V given by:

$$f(Z) = f(0) - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}, \tag{3.1}$$

where $Z^- = (Z_1, \cdots, Z_j), Z^+ = (Z_{j+1}, \cdots, Z_n)$ and (Z^-, Z^+) denotes the coordinate functions on V corresponding to the decomposition $V = V^- \oplus V^+$.

Let $Z = \sum_{\alpha=1}^n Z_\alpha e_\alpha$ be the radial vector field on V. There is a natural Euclidean scalar product on ΛV^*. Let $dv_V(Z)$ be the volume form on V. Denote by $L^2(\Lambda V^*)$ the set of the square integrable sections of ΛV^* over V. For $w_1, w_2 \in L^2(\Lambda V^*)$, set

$$\langle w_1, w_2\rangle = \int_V \langle w_1, w_2\rangle_{\Lambda V^*} dv_V(Z). \tag{3.2}$$

Let $C(V)$ be the Clifford algebra of V, i.e., the algebra generated over $\mathbb{R}$ by $e \in V$ and the commutation relations $e \cdot e' + e' \cdot e = -2\langle e, e'\rangle$ for $e, e' \in V$. Let $c(e), \widehat{c}(e)$ be the Clifford operators acting on ΛV^* defined by

$$c(e) = e^* \wedge -i_e,\ \widehat{c}(e) = e^* \wedge +i_e, \tag{3.3}$$

where $e^*\wedge$ and i_e are the standard notations for exterior and interior multiplication and e^* denotes the dual of e by the Euclidean scalar product on V. Then ΛV^* is a Clifford module.

If $X, Y \in V$, then

$$\begin{aligned} c(X)c(Y) + c(Y)c(X) &= -2\langle X, Y\rangle, \\ \widehat{c}(X)\widehat{c}(Y) + \widehat{c}(Y)\widehat{c}(X) &= 2\langle X, Y\rangle, \\ c(X)\widehat{c}(Y) + \widehat{c}(Y)c(X) &= 0. \end{aligned} \tag{3.4}$$

Let d be the exterior differential derivative acting on the smooth sections of ΛV^*, and let δ be the formal adjoint of d with respect to the Euclidean product (3.2). Denote by v the gradient of f with respect to the given Euclidean scalar product, then

$$v(Z) = -\sum_{\alpha=1}^{j} Z_\alpha e_\alpha + \sum_{\alpha=j+1}^{n} Z_\alpha e_\alpha. \tag{3.5}$$

Set

$$\begin{gathered} d_T = e^{-Tf} d \cdot e^{Tf} = d + T\, df\wedge, \quad \delta_T = e^{Tf}\delta \cdot e^{-Tf} = \delta + T\, i_v, \\ D_{T,v} = d_T + \delta_T = d + \delta + T\widehat{c}(v). \end{gathered} \tag{3.6}$$

The Witten Laplacian on Euclidean space V is by definition, the square of $D_{T,v}$. Let Δ be the standard Laplacian on V, i.e.,

$$\Delta = -\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial Z_\alpha^2}. \tag{3.7}$$

Let $e^1, \dots, e^n$ be the dual basis associated to $e_1, \dots, e_n$. Then we have the following result [19], [20, Prop. 4.9].

Proposition 3.1. *The kernel of $D^2_{T,v}$ is of one dimension and is spanned by*

$$\beta = e^{-\frac{T|Z|^2}{2}} e^1 \wedge \dots \wedge e^j. \tag{3.8}$$

Moreover, all nonzero eigenvalues of $D^2_{T,v}$ are $\geqslant 2T$.

Proof. We recall the proof for the reader's convenience.

For $e \in V$, let ∇_e denote the derivative along the vector e. It is easy to calculate the square of $D_{T,v}$,

$$\begin{aligned} D^2_{T,v} &= \Delta + T^2|Z|^2 + T\sum_{\alpha=1}^{n} c(e_\alpha)\widehat{c}(\nabla_{e_\alpha} v) \\ &= (\Delta + T^2|Z|^2 - Tn) + T\sum_{\alpha=1}^{j}\big[1 - c(e_\alpha)\widehat{c}(e_\alpha)\big] + T\sum_{\alpha=j+1}^{n}\big[1 + c(e_\alpha)\widehat{c}(e_\alpha)\big] \\ &= (\Delta + T^2|Z|^2 - Tn) + 2T\Big(\sum_{\alpha=1}^{j} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=j+1}^{n} e^\alpha \wedge i_{e_\alpha}\Big). \end{aligned} \tag{3.9}$$

The operator

$$\mathcal{L}_T = \Delta + T^2|Z|^2 - Tn \tag{3.10}$$

is the harmonic oscillator operator on V. By [13, Appendix E], we know that $\mathcal{L}_T$ is positive elliptic operator with one dimensional kernel generated by $e^{-\frac{T|Z|^2}{2}}$. Moreover, the nonzero eigenvalues of $\mathcal{L}_T$ are all greater than $2T$. It is also easy to verify that the linear operator

$$\sum_{\alpha=1}^{j} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=j+1}^{n} e^\alpha \wedge i_{e_\alpha} \tag{3.11}$$

is positive with one dimensional kernel generated by

$$e^1 \wedge \ldots \wedge e^j. \tag{3.12}$$

The proof of Proposition 3.1 is complete. $\square$

3.2. **The Witten Laplacian on $\mathbb{R}^n_+$.** Let V_1 be an $n-1$ dimensional real vector space endowed with an Euclidean scalar product. Let V_1^-, V_1^+ be two subspaces such that $V_1 = V_1^- \oplus V_1^+$ and dim $V_1^- = j$. Let $e_1, \ldots, e_{n-1}$ denote an orthonormal basis on V_1 such that V_1^- is spanned by $e_1, \ldots, e_j$. Set $V_2 = V_1 \times \mathbb{R}_+$. Denote by e_n the oriented basis of $\mathbb{R}_+$ with unit length.

Let f be a smooth function on V_2 given by:

$$f(Z) = f(0) - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2} + Z_n, \tag{3.13}$$

where $Z^- = (Z_1, \ldots, Z_j), Z^+ = (Z_{j+1}, \ldots, Z_{n-1}), (Z^-, Z^+)$ denotes the coordinate functions on V_1 corresponding to the decomposition $V_1 = V_1^- \oplus V_1^+$ and $Z_n \geqslant 0$ denotes the coordinate function on $\mathbb{R}_+$. Set $Z' = (Z^-, Z^+)$.

Let $L^2(\Lambda V_2^*)$ be the set of the square integrable sections of ΛV_2^*. We define an inner product in $L^2(\Lambda V_2^*)$ similarly to (3.2). Let $C(V_2)$ be the Clifford algebra of V_2. We still denote by $c(e), \hat{c}(e)$ the Clifford operators on ΛV_2^* for $e \in V_2$.

The gradient vector field v_1 of f with respect to the given Euclidean scalar product is now given as

$$v_1(Z) = -\sum_{\alpha=1}^{j} Z_\alpha e_\alpha + \sum_{\alpha=j+1}^{n-1} Z_\alpha e_\alpha + e_n. \tag{3.14}$$

Set

$$\begin{gathered} d_T = e^{-Tf} d \cdot e^{Tf} = d + T\ df \wedge, \quad \delta_T = e^{Tf} \delta \cdot e^{-Tf} = \delta + T\ i_{v_1}, \\ D_{T,v_1} = d_T + \delta_T = d + \delta + T\widehat{c}(v_1). \end{gathered} \tag{3.15}$$

To calculate explicitly the kernel of D^2_{T,v_1} and, more generally, to study its spectrum, we need to consider boundary conditions on ΛV_2^*, i.e., to specify the domain of a self-adjoint extension of D^2_{T,v_1}. We follow here [13, § 3.5]. Denote by $\Omega(V_2)$ the space of smooth sections of ΛV_2^* on V_2. Define the domain of the weak maximal extension of d_T by

$$\mathrm{Dom}(d_T) = \left\{ w \in L^2(\Lambda V_2^*) : d_T w \in L^2(\Lambda V_2^*) \right\} \tag{3.16}$$

where $d_T w$ is calculated in the sense of distributions. We denote by d_T^* the Hilbert space adjoint of d_T. Let e^n be the dual basis associated to e_n. Every $w \in \Omega(V_2)$ has a natural decomposition into the norm and the tangent components along V_1,

$$w = w_{\tan} + w_{\mathrm{norm}}, \tag{3.17}$$

where $w_{\tan}$ does not contain the factor e^n. Integration by parts shows

$$\begin{gathered} \mathrm{Dom}(d_T^*) \cap \Omega(V_2) = \left\{ w \in \Omega(V_2) : w_{\mathrm{norm}} = 0 \right\}, \\ d_T^* w = \delta_T w \ \text{ for } w \in \mathrm{Dom}(d_T^*) \cap \Omega(V_2)\,. \end{gathered} \tag{3.18}$$

We define then the Gaffney extension of D^2_{T,v_1} by

(3.19)
$$\begin{aligned}\mathrm{Dom}(D^2_{T,v_1}) = \big\{w \in \mathrm{Dom}(d_T) \cap \mathrm{Dom}(d^*_T) : d_T w \in \mathrm{Dom}(d^*_T)\,,\, d^*_T w \in \mathrm{Dom}(d_T)\big\}\,,\\ D^2_{T,v_1} w = d_T d^*_T w + d^*_T d_T w \ \text{ for } w \in \mathrm{Dom}(D^2_{T,v_1})\,.\end{aligned}$$

This is a self-adjoint operator, see [13, Prop. 3.1.2]. The smooth forms in the domain of D^2_{T,v_1} satisfy the following boundary conditions

$$\mathrm{Dom}(D^2_{T,v_1}) \cap \Omega(V_2) = \Big\{w \in \Omega(V_2), \begin{aligned} w_{\mathrm{norm}} &= 0\\ \big(d_T w\big)_{\mathrm{norm}} &= 0\end{aligned} \quad \text{on } V_1\Big\}. \tag{3.20}$$

If we rewrite $w \in \Omega(V_2)$ as

$$w(Z', Z_n) = w_1(Z', Z_n) + e^n \wedge w_2(Z', Z_n),\ \ Z' \in V_1, Z_n \in \mathbb{R}_+, \tag{3.21}$$

where w_1 does not contain the factor e^n, then we have

Lemma 3.2. *The boundary conditions*

$$w_{\mathrm{norm}} = 0,\ \ (d_T w)_{\mathrm{norm}} = 0 \text{ on } V_1 \tag{3.22}$$

are equivalent to

$$\frac{\partial w_1}{\partial Z_n}(Z', 0) + T w_1(Z', 0) = 0 \text{ and } w_2(Z', 0) = 0. \tag{3.23}$$

Proof. The proof is straightforward and is left to the reader. □

Let $e^1, \dots, e^{n-1}$ be the dual basis of $e_1, \dots, e_{n-1}$. We denote by Δ' (resp. Δ) the standard Laplacian on V_1 (resp. V_2), i.e.,

$$\Delta' = -\sum_{\alpha=1}^{n-1} \frac{\partial^2}{\partial Z_\alpha^2} \quad \Big(\text{resp. } \Delta = -\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial Z_\alpha^2}\Big). \tag{3.24}$$

Proposition 3.3. *The kernel of D^2_{T,v_1}, with the domain given as in* (3.19), *is one dimensional and is spanned by*

$$\beta_{T,v_1} = e^{-\frac{T}{2}|Z'|^2 - T Z_n} e^1 \wedge \dots \wedge e^j. \tag{3.25}$$

Moreover, all nonzero eigenvalues of D^2_{T,v_1} are $\geqslant 2T$.

Proof. We adapt the proof from [20, Prop. 4.9]. We denote by ∇_e the derivative along a vector $e \in V_2$. It is easy to calculate the square of D_{T,v_1}:

$$
\begin{aligned}
D^2_{T,v_1} =& \Delta + T^2\Big(|Z'|^2 + 1\Big) + T\sum_{\alpha=1}^{n} c(e_\alpha)\widehat{c}\big(\nabla_{e_\alpha} v_1\big) \\
=& \Big(\Delta' + T^2|Z'|^2 - T(n-1)\Big) + \Big(-\frac{\partial^2}{\partial Z_n^2} + T^2\Big) \\
& + T\sum_{\alpha=1}^{j}\big[1 - c(e_\alpha)\widehat{c}(e_\alpha)\big] + T\sum_{\alpha=j+1}^{n-1}\big[1 + c(e_\alpha)\widehat{c}(e_\alpha)\big] \\
=& \Big(\Delta' + T^2|Z'|^2 - T(n-1)\Big) + \Big(-\frac{\partial^2}{\partial Z_n^2} + T^2\Big) \\
& + 2T\Big(\sum_{\alpha=1}^{j} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=j+1}^{n-1} e^\alpha \wedge i_{e_\alpha}\Big).
\end{aligned}
\tag{3.26}
$$

Since the three operators in parentheses on the right side of the third equality in (3.26) commute with each other, we can calculate the kernels of D^2_{T,v_1} via the method of separating variables. As in Proposition 3.1, the kernel of the operator

$$
\mathcal{L}'_T = \Delta' + T^2|Z'|^2 - T(n-1) \tag{3.27}
$$

is one dimensional and generated by $e^{-\frac{T|Z'|^2}{2}}$. Besides, its nonzero eigenvalues are all greater than $2T$. It is easy to verify that the linear operator

$$
\sum_{\alpha=1}^{j} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=j+1}^{n-1} e^\alpha \wedge i_{e_\alpha} \tag{3.28}
$$

restricted to $\mathrm{Dom}(D^2_{T,v_1})$ is positive and has one dimensional kernel generated by

$$
e^1 \wedge \ldots \wedge e^j. \tag{3.29}
$$

Therefore, the elements of the kernel of D^2_{T,v_1} take the form

$$
g(Z_n)e^{-\frac{T|Z'|^2}{2}} e^1 \wedge \ldots \wedge e^j, \tag{3.30}
$$

where $g(Z_n)$ is a smooth function in Z_n. Combining (3.23) and (3.26), we find that the smooth function $g(Z_n)$ satisfies the following conditions:

$$
g'(0) + Tg(0) = 0, \quad -g''(Z_n) + T^2 g(Z_n) = 0. \tag{3.31}
$$

Thus, $g(Z_n) = e^{-TZ_n}$ up to multiplicative constant. Set

$$
R_T = -\frac{\partial^2}{\partial Z_n^2} + T^2 \tag{3.32}
$$

with domain given by

$$
\mathrm{Dom}(R_T) = \big\{h \in C_0^\infty(\mathbb{R}_+),\ h'(0) + Th(0) = 0\big\}. \tag{3.33}
$$

One verifies directly via integration by parts that R_T is positive on $\mathrm{Dom}(R_T)$, so R_T has a self-adjoint positive extension (the Friedrichs extension) still denoted by R_T. Denote

by $\lambda_1(P)$ the first nonzero eigenvalues of a positive operator P. By min-max principle [13, (C.3.3)] and the fact that R_T and (3.28) are positive operators, we deduce that

$$\lambda_1(D^2_{T,v_1}) \geqslant \lambda_1(\mathcal{L}'_T) = 2T. \tag{3.34}$$

The proof of Proposition 3.3 is complete. □

4. The Witten instanton complex

Let g^{TM} be a metric on TM such that if $p \in C^j(f) \cup C^j_-(f|_{\partial M})$, in the coordinates $x = (x_1, \dots, x_n)$ in the conditions (2) and (3) of Section 2,

$$g^{TM} = \sum_{\alpha=1}^{n} dx_\alpha^2 \quad \text{on } U_p. \tag{4.1}$$

Let ∇^{TM} be the Levi-Civita connection associated to the metic g^{TM}, and let dv_M be the density (or Riemannian volume form) on M, i.e., dv_M is a smooth section of the line bundle $\Lambda^n(T^*M) \otimes o(TM)$ (cf. [1, p. 29], [6, p. 88]). Denote by $\Omega^i(M)$ the space of smooth differential i-forms on M. Set $\Omega(M) = \oplus_{i=0}^n \Omega^i(M)$. We denote by $L^2\Omega(M)$ the space of square integrable sections of $\Lambda(T^*M)$ over M. For $w_1, w_2 \in L^2\Omega(M)$, set

$$\langle w_1, w_2 \rangle = \int_M \langle w_1, w_2 \rangle (x) dv_M(x). \tag{4.2}$$

We denote by $\|\cdot\|$ the norm on $L^2\Omega(M)$ induced by (4.2).

Let d be the exterior differential derivative on $\Omega(M)$, and let δ be the formal adjoint of d with respect to the metric (4.2). Set

$$d_T = e^{-Tf} d \cdot e^{Tf}, \quad \delta_T = e^{Tf} \delta \cdot e^{-Tf}. \tag{4.3}$$

The deformed de Rham operator D_T is given by

$$D_T = d_T + \delta_T. \tag{4.4}$$

The Witten Laplacian on manifolds is defined by

$$D_T^2 = (d_T + \delta_T)^2 = d_T\delta_T + \delta_T d_T. \tag{4.5}$$

Define the domain of the weak maximal extension of d_T by

$$\mathrm{Dom}(d_T) = \left\{ w \in L^2\Omega(M), d_T w \in L^2\Omega(M) \right\} \tag{4.6}$$

where $d_T w$ is calculated in the sense of distributions. We denote by d_T^* the Hilbert space adjoint of d_T. Every smooth differential form w has a natural decomposition into the norm and the tangent components along ∂M,

$$w = w_{\mathrm{tan}} + w_{\mathrm{norm}}, \tag{4.7}$$

where w_{tan} does not contain the factor ν. Integration by parts shows

$$\begin{gathered} \mathrm{Dom}(d_T^*) \cap \Omega(M) = \left\{ w \in \Omega(M) : w_{\mathrm{norm}} = 0 \right\}, \\ d_T^* w = \delta_T w \ \text{ for } w \in \mathrm{Dom}(d_T^*) \cap \Omega(M)\,. \end{gathered} \tag{4.8}$$

By (4.8), the domain of the extension $D_T = d_T + d_T^*$ of deformed de Rham operator is:

$$\mathrm{Dom}(D_T) \cap \Omega(M) = \left\{ w \in \Omega(M), w_{\mathrm{norm}} = 0 \ \text{ on } \partial M \right\}. \tag{4.9}$$

We define the self-adjoint extension of D_T^2 as in (3.19) by $D_T^2 = d_T d_T^* + d_T^* d_T$. Then

$$\mathrm{Dom}(D_T^2) \cap \Omega(M) = \Big\{ w \in \Omega(M), \begin{array}{r} w_{\mathrm{norm}} = 0 \\ \big(d_T w\big)_{\mathrm{norm}} = 0 \end{array} \quad \text{on } \partial M \Big\}. \tag{4.10}$$

Following the argument of [20, Prop. 5.5], one easily gets Morse inequalities (1.6) granted the following Proposition holds.

Proposition 4.1. *For any $C_0 > 0$, there exists $T_0 > 0$ such that when $T \geqslant T_0$, the number of eigenvalues in $[0, C_0)$ of $D_T^2|_{\mathrm{Dom}(D_T^2)\cap\Omega^j(M)}$ equals $c_j + p_j$.*

We postpone the proof of Proposition 4.1 to Section 5.4. We prove now Corollary 1.4 by using Proposition 4.1.

Proof of Corollary 1.4. Let $F_{T,j}^{C_0}$ denote the $(c_j + p_j)$-dimensional vector space generated by the eigenspaces of $D_T^2|_{\mathrm{Dom}(D_T^2)\cap\Omega^j(M)}$ associated to the eigenvalues lying in $[0,\, C_0)$. Since $d_T D_T^2 = D_T^2 d_T$, one verifies directly that $d_T(F_{T,j}^{C_0}) \subset F_{T,j+1}^{C_0}$. Then we have the following complex:

$$(F_{T,\bullet}^{C_0}, d_T) : 0 \longrightarrow F_{T,0}^{C_0} \longrightarrow F_{T,1}^{C_0} \longrightarrow \ldots \to F_{T,n}^{C_0} \longrightarrow 0. \tag{4.11}$$

By Hodge Theorem in the finite dimensional case, the j-th cohomology group of the above complex is isomorphic to $\mathrm{Ker}\big(D_T^2|_{\mathrm{Dom}(D_T^2)\cap\Omega^j(M)}\big)$, which is again by Hodge Theorem isomorphic to the j-th cohomology group of the deformed de Rham complex $(\Omega^\bullet(M), d_T)$. It is a consequence of (4.3) that the j-th cohomology group of the deformed de Rham complex $(\Omega^\bullet(M),\, d_T)$ is isomorphic to the j-th cohomology group of the de Rham complex $(\Omega^\bullet(M), d)$. Then the inequalities (1.6) follow from standard algebraic techniques ([13, Lemma 3.2.12]). □

Remark 4.2. We can also obtain the inequalities (1.6) immediately by combining Theorem 1.3 and [12, Th. A].

Proof of Corollary 1.5. By Poincaré duality theorem for non-orientable manifolds,

$$H_{dR}^\bullet(M, \mathbb{R}) \simeq H_{dR}^{n-\bullet}(M, \partial M; o(TM)). \tag{4.12}$$

Then the inequalities (1.7) follow immediately from the inequalities (1.6) and the isomorphism (4.12). □

5. Proof of Proposition 4.1 and Theorem 1.3

The organization of the Section is as follows. In Section 5.1, we obtain a basic estimate for the deformed de Rham operator which allows to localize our problem (i.e., study the eigenvectors with small eigenvalues of the Witten Laplacian) to some neighborhood of critical points in $C(f)$ and $C_-(f|_{\partial M})$. Section 5.2 is devoted to the local behavior of the Witten Laplacian around critical points in $C(f)$ and $C_-(f|_{\partial M})$. In Section 5.3, we get a decomposition of the deformed de Rham operator and establish estimates of its components. We also prove Theorem 1.1 and Theorem 1.2 there. Section 5.4 is devoted to the proof of Proposition 4.1 and Theorem 1.3.

5.1. **Localization of the lower part of the spectrum of the Witten Laplacian.** Choose a small enough such that all $B(p,a)$'s are disjoint for $p \in C(f) \cup C_-(f|_{\partial M})$, and each $B(p,a)$ lies in the interior of M for $p \in C(f)$. Denote by U the union of all $B(p,a)$'s for $p \in C(f) \cup C_-(f|_{\partial M})$.

Proposition 5.1. *There exist constants $a_4 > 0, T_3 > 0$ such that for any $s \in \mathrm{Dom}(D_T)$ with $\mathrm{supp}(s) \subset M \backslash U$ and $T \geqslant T_3$, we have*

$$\big\|D_T s\big\| \geqslant a_4 T \big\|s\big\|. \tag{5.1}$$

Proof. We adapt our proof from [15, pp. 29-30]. Note that the Green formula holds also on non-orientable manifolds, see [18, Chapter 2, Th. 2.1]. It is a consequence of (4.5) and the Green's formula that

$$\big\|D_T s\big\|^2 = \big\langle d_T s, d_T s\big\rangle + \big\langle \delta_T s, \delta_T s\big\rangle. \tag{5.2}$$

Let ∇f denote the gradient field of f with respect to the metric g^{TM}. Since $s_{\mathrm{norm}} = 0$, one verifies directly from the Green's formula that

$$\begin{aligned}\big\langle d_T s, d_T s\big\rangle + \big\langle \delta_T s, \delta_T s\big\rangle =& \big\langle ds, ds\big\rangle + \big\langle \delta s, \delta s\big\rangle + T^2 \big\langle |\nabla f|^2 s, s\big\rangle \\ &+ T\big\langle Qs, s\big\rangle + T\int_{\partial M} \big\langle (\nu f) s_{\tan}, s_{\tan}\big\rangle d_{\partial M}(x),\end{aligned} \tag{5.3}$$

where $dv_{\partial M}$ denotes the density on ∂M induced by dv_M, ν outward normal field and Q is an endomorphism of $\Omega(M)$ given by

$$Q = \sum_{i=1}^{n} c(e_i)\hat{c}\big(\nabla^{TM}_{e_i}(\nabla f)\big). \tag{5.4}$$

If $\mathrm{supp}(s) \cap \partial M = \emptyset$, then

$$\int_{\partial M} \big\langle (\nu f) s_{\tan}, s_{\tan}\big\rangle d_{\partial M}(x) = 0. \tag{5.5}$$

Then (5.1) follows immediately from (5.2), (5.3) and (5.5). Suppose now $\mathrm{supp}(s) \cap \partial M \neq \emptyset$. Clearly,

$$-(vf)(x) < |\nabla f|(x), \quad \text{for any } x \in \ \mathrm{supp}(s) \cap \partial M. \tag{5.6}$$

Choose $\eta > 0$ small enough so that

$$-(vf)(x) < (1-\eta)|\nabla f|(x), \quad \text{for any } x \in \ \mathrm{supp}(s) \cap \partial M. \tag{5.7}$$

Locally it is possible to construct a function $\tilde{f}$ such that

$$\begin{aligned}\big|\nabla \tilde{f}\big| &= \big|\nabla f\big|, \quad \text{in} \quad \mathrm{supp}(s); \\ \big|\nabla \tilde{f}\big| &= -\nu \tilde{f}, \quad \text{in} \quad \mathrm{supp}(s) \cap \partial M.\end{aligned} \tag{5.8}$$

Let $\tilde{V}$ be an open neighborhood of ∂M such that the equations (5.8) are solvable on $V_r \cap \mathrm{supp}(s)$ with $V_r = \tilde{V} \times [0, r)$. By a partition of unity argument, we now assume that $s \in \mathrm{Dom}(D_T)$ with $\mathrm{supp}(s) \subset V_r$ and the solution $\tilde{f}$ of the equations (5.8) is a smooth function on M. Then (5.7) and (5.8) imply

$$T\int_{\partial M} \big\langle (\nu f) s_{\tan}, s_{\tan}\big\rangle d_{\partial M}(x) \geqslant T(1-\eta)\int_{\partial M} \big\langle (\nu \tilde{f}) s_{\tan}, s_{\tan}\big\rangle d_{\partial M}(x). \tag{5.9}$$

Applying the equality (5.3) to the smooth function $\tilde{f}$, we find

$$\begin{aligned} & T\int_{\partial M}\big\langle(\nu f)s_{\tan},s_{\tan}\big\rangle d_{\partial M}(x) \\ & \geqslant -(1-\eta)\Big[\big\langle ds,ds\big\rangle+\big\langle \delta s,\delta s\big\rangle+T^2\big\langle|\nabla f|^2 s,s\big\rangle+C_1T\|s\|^2\Big], \end{aligned} \tag{5.10}$$

where $C_1>0$ is independent of s. Substituting (5.10) into (5.3), we obtain the estimate (5.1). The proof of Proposition 5.1 is complete. □

In view of Proposition 5.1, the eigenvectors with small eigenvalues of the Witten Laplacian D_T^2 "concentrate" for T large around the critical points in $C(f)$ and $C_-(f|_{\partial M})$.

5.2. **Local behavior of the Witten Laplacian around critical points in $C(f)$ and $C_-(f|_{\partial M})$.** If $p\in C^j(f)$, then there exists a coordinate system (x,U_p) such that for any $x\in U_p$,

$$f(x)=f(p)-\frac{x_1^2}{2}-\ldots-\frac{x_j^2}{2}+\frac{x_{j+1}^2}{2}+\ldots+\frac{x_n^2}{2}. \tag{5.11}$$

Over U_p, set $e_k=\frac{\partial}{\partial x_k}$ for $k=1,\ldots,n$. Then the dual basis $e^k=dx_k$ for all k. From (3.1), (3.6), (4.4) and (5.11), we have

$$D_T|_{U_p}=D_{T,v}|_{U_p}. \tag{5.12}$$

Set

$$\begin{aligned} \mathcal{L}_T &=-\sum_{\alpha=1}^{n}\frac{\partial^2}{\partial x_\alpha^2}+T\sum_{\alpha=1}^{n}x_\alpha^2-Tn, \\ \mathcal{K}_T &=2T\Big(\sum_{\alpha=1}^{j}i_{e_\alpha}e^\alpha\wedge+\sum_{\alpha=j+1}^{n}e^\alpha\wedge i_{e_\alpha}\Big) \end{aligned} \tag{5.13}$$

Then (3.9) and (5.12) imply that

$$D_T^2=\mathcal{L}_T+\mathcal{K}_T \tag{5.14}$$

holds throughout U_p. Denote by $L^2\Omega(\mathbb{R}^n)$ the space of square integrable differential forms on $\mathbb{R}^n$. By Proposition 3.1, we have

Proposition 5.2. *For any $T>0$, the operator $\mathcal{L}_T+\mathcal{K}_T$ acting on $L^2\Omega(\mathbb{R}^n)$ is an essentially self-adjoint positive operator. Its kernel is one dimensional and is spanned by*

$$\beta_T=e^{-\frac{T}{2}|x|^2}e^1\wedge\ldots\wedge e^j. \tag{5.15}$$

Moreover, all nonzero eigenvalues of $\mathcal{L}_T+\mathcal{K}_T$ are $\geqslant 2T$.

If $p\in C_-^j(f|_{\partial M})$, then there exists a coordinate system (x,U_p) such that for any $x\in U_p$,

$$f(x)=f(p)-\frac{x_1^2}{2}-\ldots-\frac{x_j^2}{2}+\frac{x_{j+1}^2}{2}+\ldots\frac{x_{n-1}^2}{2}+x_n. \tag{5.16}$$

On U_p, set $e_k=\frac{\partial}{\partial x_k}$ for $k=1,\ldots,n$. Then the dual basis $e^k=dx_k$ for all k. In view of (3.13), (3.15), (4.4) and (5.16), we have

$$D_T|_{U_p}=D_{T,v_1}|_{U_p}. \tag{5.17}$$

Set

$$
\begin{aligned}
\mathcal{L}'_T &= -\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial x_\alpha^2} + T\sum_{\alpha=1}^{n-1} x_\alpha^2 - T(n-1) + T^2, \\
\mathcal{K}'_T &= 2T\Big(\sum_{\alpha=1}^{j} i_{e_\alpha} e^\alpha \wedge + \sum_{\alpha=j+1}^{n-1} e^\alpha \wedge i_{e_\alpha}\Big).
\end{aligned} \tag{5.18}
$$

Combining (3.26) and (5.17), we find that on U_p,

$$
D_T^2 = \mathcal{L}'_T + \mathcal{K}'_T. \tag{5.19}
$$

Let $\Omega(\mathbb{R}^n_+)$ be the space of smooth differential sections on $\mathbb{R}^n_+$. Consider the self-adjoint extension of (5.19) defined as in (3.16)-(3.19). Thus

$$
\operatorname{Dom}\big(\mathcal{L}'_T + \mathcal{K}'_T\big) \cap \Omega(\mathbb{R}^n_+) = \Big\{ w \in \Omega(\mathbb{R}^n_+), \begin{aligned} w_{\mathrm{norm}} &= 0 \\ \big(d_T w\big)_{\mathrm{norm}} &= 0 \end{aligned} \quad \text{on } \partial\mathbb{R}^n_+ \Big\}. \tag{5.20}
$$

By Proposition 3.3, we have

Proposition 5.3. *For any $T > 0$, the self-adjoint extension of $\mathcal{L}'_T + \mathcal{K}'_T$, given as in* (3.16)-(3.19) *is a positive operator. Its kernel is one dimensional and is spanned by*

$$
\xi_T = e^{-\frac{T}{2}|x'|^2 - Tx_n} e^1 \wedge \ldots \wedge e^j. \tag{5.21}
$$

Moreover, all nonzero eigenvalues of $\mathcal{L}'_T + \mathcal{K}'_T$ are $\geqslant 2T$.

5.3. **A decomposition of the deformed de Rham operator D_T.** Let $\gamma : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\gamma(x) = 1$ if $|x| \leqslant a$ and that $\gamma(x) = 0$ if $|x| \geqslant 2a$. For any $p \in C^j(f)$ and $q \in C^j_-(f|_{\partial M})$, set

$$
\begin{aligned}
\alpha_{p,T} &= \int_{U_p} \gamma(|x|)^2 e^{-T|x|^2} dx_1 \wedge \ldots \wedge dx_n, \\
\alpha_{q,T} &= \int_{U_q} \gamma(|x|)^2 e^{-T|x'|^2 - 2Tx_n} dx_1 \wedge \ldots \wedge dx_n.
\end{aligned} \tag{5.22}
$$

Clearly, there exists $c > 0$ such that as $T \to +\infty$,

$$
\begin{aligned}
\alpha_{p,T} &= \Big(\frac{\pi}{T}\Big)^{\frac{n}{2}} + O(e^{-cT}), \\
\alpha_{q,T} &= \frac{1}{2T}\Big(\frac{\pi}{T}\Big)^{\frac{n-1}{2}} + O(e^{-cT}).
\end{aligned} \tag{5.23}
$$

Set

$$
\begin{aligned}
\rho_{p,T} &= \frac{\gamma(|x|)}{\sqrt{\alpha_{p,T}}} e^{-\frac{T}{2}|x|^2} dx_1 \wedge \ldots \wedge dx_j, \\
\rho_{q,T} &= \frac{\gamma(|x|)}{\sqrt{\alpha_{q,T}}} e^{-\frac{T}{2}|x'|^2 - Tx_n} dx_1 \wedge \ldots \wedge dx_j.
\end{aligned} \tag{5.24}
$$

Let E_T^j be the direct sum of the vector spaces generated by all $\rho_{p,T}$'s and $\rho_{q,T}$'s with $p \in C^j(f)$ and $q \in C^j_-(f|_{\partial M})$. Set $E_T = \oplus_{j=0}^n E_T^j$. Clearly,

$$\dim E_T = \sum_{j=0}^{n} \big(c_j + p_j\big). \tag{5.25}$$

Take $E_T^\perp$ as the orthogonal complement of E_T in $\mathrm{Dom}(D_T)$, then $\mathrm{Dom}(D_T)$ has an orthogonal splitting:

$$\mathrm{Dom}(D_T) = E_T \oplus E_T^\perp. \tag{5.26}$$

Let $p_1, p_1^\perp$ denote the orthogonal projections from $\mathrm{Dom}(D_T)$ onto E_T and $E_T^\perp$, respectively. Also we have another orthogonal splitting about E_T in $L^2\Omega(M)$:

$$L^2\Omega(M) = E_T \oplus F_T, \tag{5.27}$$

where F_T is the orthogonal complement of E_T in $L^2\Omega(M)$. Then $E_T^\perp \subset F_T$. Denote by $p_2, p_2^\perp$ the orthogonal projections from $L^2\Omega(M)$ onto E_T and F_T, respectively. Following Bismut-Lebeau [4, §9], we decompose the deformed de Rham operator D_T according to the splittings (5.26) and (5.27):

$$\begin{aligned} D_{T,1} &= p_2 D_T p_1, \quad D_{T,2} = p_2 D_T p_1^\perp, \\ D_{T,3} &= p_2^\perp D_T p_1 \quad D_{T,4} = p_2^\perp D_T p_1^\perp. \end{aligned} \tag{5.28}$$

Then

$$D_T = D_{T,1} + D_{T,2} + D_{T,3} + D_{T,4}. \tag{5.29}$$

Denote by $\mathbf{H}^1(M)$ the first Sobolev space with respect to a (fixed) Sobolev norm on $\Omega(M)$. The analogues of the estimates [20, Prop. 5.6] still hold for the operators $D_{T,j}$:

Proposition 5.4. *(1) For any $T > 0$,*

$$D_{T,1} = 0; \tag{5.30}$$

(2) There exist positive constants b_1, b_2 and T_4 such that for any $s \in E_T^\perp \cap \mathbf{H}^1(M)$, $s' \in E_T$ and any $T \geqslant T_4$, one has

$$\begin{aligned} \big\|D_{T,2} s\big\| &\leqslant b_1 e^{-b_2 T}\big\|s\big\|, \\ \big\|D_{T,3} s'\big\| &\leqslant b_1 e^{-b_2 T}\big\|s'\big\|. \end{aligned} \tag{5.31}$$

(3) There exists constant $b_3 > 0$ and $T_5 > 0$, such that for any $s \in E_T^\perp \cap \mathbf{H}^1(M)$ and any $T \geqslant T_5$, one has

$$\big\|D_{T,4} s\big\| \geqslant b_3 T \big\|s\big\|. \tag{5.32}$$

Proof. The proof is similar to [20, Prop. 5.6] except for the estimate of $D_{T,3}$. Compared to [4, (9.17)] and [20, (5.19)], the operator $D_{T,3}$ is no longer the formal adjoint of $D_{T,2}$ due to the fact that the image of D_T acting on $\mathrm{Dom}(D_T)$ does not necessarily lie in $\mathrm{Dom}(D_T)$. We prove the estimate of $D_{T,3}$ directly. From (5.24), Proposition 5.2 and Proposition 5.3, we have

$$D_T(\rho_{p,T}) = \frac{c(\nabla\gamma)}{\sqrt{\alpha_{p,T}}} e^{-\frac{T}{2}|x|^2} dx_1 \wedge \ldots \wedge dx_j \tag{5.33}$$

and

$$D_T(\rho_{q,T}) = \frac{c(\nabla\gamma)}{\sqrt{\beta_{q,T}}} e^{-\frac{T}{2}|x'|^2 - Tx_n} dx_1 \wedge \ldots \wedge dx_j, \tag{5.34}$$

where $c(\nabla\gamma)$ denotes the endomorphism on $\Omega(M)$ given as

$$c(\nabla\gamma) = \sum_{i=1}^{n} (e_i\gamma)c(e_i). \tag{5.35}$$

Then (5.33) and (5.34) imply the second inequality in (5.31). The rest of the proof is similar to [20, Prop. 5.6]. □

Proof of Theorem 1.1 and Theorem 1.2. By the min-max principle [13, (C.3.3)],

$$\lambda_k(T) = \inf_{\substack{F\subset \mathrm{Dom}(D_T^2),\\ \dim F = k}} \ \sup_{\substack{s\in F,\\ \|s\|=1}} \left\langle D_T^2 s, s\right\rangle. \tag{5.36}$$

Take $F = E_T^j$, then $\dim F = c_j + p_j$. It is a consequence of (5.30) and (5.31) that for every $s \in F$,

$$\left\|D_T s\right\|^2 = \left\|D_{T,3} s\right\|^2 \leqslant b_1^2 e^{-2b_2 T}\left\|s\right\|^2. \tag{5.37}$$

Then (1.3) follows immediately from (5.36) and (5.37). Suppose now that F is a (c_j+p_j+1)-dimensional subspace of $\mathrm{Dom}(D_T^2)\cap\Omega^j(M)$. Clearly $F\cap E_T^\perp \neq \{0\}$. Let $s\in F\cap E_T^\perp$ be a nonzero element. Then (5.31) yields

$$\left\|D_T s\right\|^2 = \left\|D_{T,2} s\right\|^2 + \left\|D_{T,4} s\right\|^2 \geqslant \left\|D_{T,4} s\right\|^2 \geqslant b_3^2 T^2 \left\|s\right\|^2. \tag{5.38}$$

Relations (5.36) and (5.38) imply immediately (1.2). □

5.4. **Proof of Proposition 4.1 and Theorem 1.3.** Denote by $F_T^{C_0}$ the finite dimensional vector space consisting of eigenspaces of $D_T^2|_{\mathrm{Dom}(D_T^2)}$ associated to the eigenvalues lying in $[0, C_0)$, i.e., $F_T^{C_0} = \bigoplus_{j=0}^n F_{T,j}^{C_0}$. Denote by $P_T^{C_0}$ the orthogonal projection operator from E to $F_T^{C_0}$. Since D_T^2 preserves the degree of $\Omega^\bullet(M)$, the projection $P_T^{C_0}$ maps $\Omega^j(M)$ onto $F_{T,j}^{C_0}$. Let J_T be the linear map from C^j to E_T^j by sending $[p]^*$ to $\rho_{p,T}$ for $p\in C^j(f)\cup C_-^j(f|_{\partial M})$. Set $e_T : C^j \to F_{T,j}^{C_0}$ by $e_T = P_T^{C_0} J_T$. In view of Proposition 5.4, the following result ([20, Th. 6.7]) still holds.

Lemma 5.5. *There exists $c>0$ such that as $T\to\infty$, for any $s\in C^j$,*

$$\big(e_T - J_T\big)s = O(e^{-cT})\big\|s\big\| \ \text{ uniformly on } M. \tag{5.39}$$

In particular, e_T is an isomorphism when T is large enough.

Proof. The proof is similar to that of [20, Th. 6.7]. □

Proof of Proposition 4.1. From Lemma 5.5, there exists $T_6>0$ such that when $T\geqslant T_6$,

$$\dim F_T^{C_0} \geqslant \dim E_T. \tag{5.40}$$

We next carry on nearly word by word as in [20, pp. 86-88] to get

$$\dim F_{T,j}^{C_0} = c_j + p_j. \tag{5.41}$$

Then the proof of Proposition 4.1 is complete. □

Proof of Theorem 1.3. Substituting (1.4) into (1.5), we get for $\alpha \in F^{C_0}_{T,j}$,

$$P_{\infty,T}(\alpha) = \sum_{p \in C^j(f) \cup C^j_-(f|_{\partial M})} [p]^* \int_{\overline{W^u(p)}} e^{Tf}\alpha. \tag{5.42}$$

Note that $P_{\infty,T}$ is a chain homomorphism, i.e., $P_{\infty,T} d_T = \partial P_{\infty,T}$. Indeed, the proof of [11, Prop. 6] (in the boundaryless case) goes through also for $P_{\infty,T}$ defined in (1.5); we just apply the Stokes formula as in [11, Prop. 6] also for the closure of unstable manifolds $\overline{W^u(p)}$ for $p \in C_-(f|_{\partial M})$. Define $\mathcal{F} \in \mathrm{End}(C^j)$ by sending $[p]^*$ to $f(p) \cdot [p]^*$ for $p \in C^j(f)$ and sending $[q]^*$ to $\big(f(q) + \frac{1}{2T}\ln 2\pi\big)[q]^*$ for $q \in C^j_-(f|_{\partial M})$. Set $\mathcal{N} \in \mathrm{End}(C^j)$ by taking $[p]^*$ to $j \cdot [p]^*$ for $p \in C^j(f)$ and taking $[q]^*$ to $(j - \frac{1}{2}) \cdot [q]^*$ for $q \in C^j_-(f|_{\partial M})$. From (2.5), (2.6) and Proposition 5.4, we get the following analogue of [20, Th. 6.9]: there exists $c > 0$ such that as $T \to \infty$,

$$P_{\infty,T} e_T = e^{T\mathcal{F}} (\frac{\pi}{T})^{\frac{\mathcal{N}}{2} - \frac{n}{4}} \big(1 + O(e^{-cT})\big) \tag{5.43}$$

In particular, $P_{\infty,T}$ is an isomorphism when T is large enough. Since $P_{\infty,T}$ is a chain homomorphism, it induces an isomorphism between the cohomology groups of the two complexes. We finish the proof of Theorem 1.3. □

Remark 5.6. Given the Morse function f, the Morse-Smale complex $(C^\bullet, \partial)$ defined as in (2.7) and (2.8) depends on the vector field X. However, by [12, §2.3, Prop.], the homology of $(C^\bullet, \partial)$ is independent of the choice of the pseudo-gradient vector field. Therefore, the Witten complex $(F^{C_0}_{T,\bullet}, d_T)$ is quasi-isomorphic to all Morse-Smale complex constructed by Laudenbach in [12].

Remark 5.7. For the relative boundary case, our strategy proceeds as follows. We first find a vector field Y on M satisfying the conditions (1)–(5) in Section 2 except that the sets $C^j_-(f|_{\partial M})$, $C_-(f|_{\partial M})$ there should be replaced by the sets $C^{j-1}_+(f|_{\partial M})$ and $C_+(f|_{\partial M})$, respectively, and the condition (4) should read as
(4)′ if $p \in C^{j-1}_+(f|_{\partial M})$, there are coordinates $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ on some neighborhood U_p of p such that on U_p,

$$f(x) = f(p) - \frac{x_1^2}{2} - \ldots - \frac{x_{j-1}^2}{2} + \frac{x_j^2}{2} + \ldots + \frac{x_{n-1}^2}{2} - x_n \tag{5.44}$$

and

$$Y = \sum_{i=1}^{j-1} x_i \frac{\partial}{\partial x_i} - \sum_{i=j}^{n-1} x_i \frac{\partial}{\partial x_i} + x_n \frac{\partial}{\partial x_n}. \tag{5.45}$$

Then the corresponding Thom-Smale complex is derived by replacing the sets $C^j_-(f|_{\partial M})$ in (2.7), $C^{j+1}_-(f|_{\partial M})$ in (2.8) by the sets $C^{j-1}_+(f|_{\partial M})$ and $C^j_+(f|_{\partial M})$, respectively. On the other hand, we consider different boundary conditions for D_T^2, that is, a different self-adjoint extension. We start with the weak maximal extension of δ_T,

$$\mathrm{Dom}(\delta_T) = \big\{w \in L^2\Omega(M), \delta_T w \in L^2\Omega(M)\big\} \tag{5.46}$$

where $\delta_T w$ is calculated in the sense of distributions. We denote by δ_T^* the Hilbert space adjoint of δ_T. Hence

$$\begin{aligned} \operatorname{Dom}(\delta_T^*) \cap \Omega(M) &= \left\{ w \in \Omega(M) : w_{\tan} = 0 \right\}, \\ \delta_T^* w = d_T w \ &\text{ for } w \in \operatorname{Dom}(\delta_T^*) \cap \Omega(M)\,. \end{aligned} \tag{5.47}$$

The domain of the extension $D_T = \delta_T^* + \delta_T$ of deformed de Rham operator $d_T + \delta_T$ is $\operatorname{Dom}(\delta_T^*) \cap \operatorname{Dom}(\delta_T)$. By (5.47),

$$\operatorname{Dom}(D_T) \cap \Omega(M) = \left\{ w \in \Omega(M), w_{\tan} = 0 \ \text{ on } \partial M \right\}. \tag{5.48}$$

We define the Gaffney estension of D_T^2 as in (3.19) by $D_T^2 = \delta_T^* \delta_T + \delta_T \delta_T^*$. Then

$$\operatorname{Dom}(D_T^2) \cap \Omega(M) = \left\{ w \in \Omega(M), \begin{array}{r} w_{\tan} = 0 \\ \left(\delta_T w\right)_{\tan} = 0 \end{array} \ \text{ on } \partial M \right\}. \tag{5.49}$$

The analogues of Theorems 1.1 and 1.2 are obtained simply by replacing the number p_j in the expressions (1.2) and (1.3) by the number q_{j-1}. Then we obtain the corresponding Witten instanton complex. Moreover, the chain morphism between the Witten instanton complex and the Thom-Smale complex is constructed as in (1.4) and (1.5) except that the set $C_-(f|_{\partial M})$ in (1.4) should be replaced by the set $C_+(f|_{\partial M})$. This morphism is an isomorphism for T large enough. As a by-product, we obtain the inequalities (1.7).

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